More on Shortest and Equal Tails Confidence Intervals

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Abstract

An interesting topic in mathematical statistics is that of the construction of the confidence intervals. Two kinds of intervals which are both based on the method of the pivotal quantity are a) the Shortest Confidence Interval (SCI) and b) the Equal Tails Confidence Intervals (ETCI). The aim of this paper is i) to clarify and comment on the finding of such intervals, ii) to investigate the relation between the two kinds of intervals, iii) to point out that the existence of confidence intervals with the shortest length do not always exist, even when the distribution of the pivotal quantity is symmetric and finally iv) to give similar results when the Bayes approached is used. We believe that all these will contribute to in classroom presentation of the topic to the graduate and postgraduate students.

Key Words: Pivotal quantity; Monotonicity; Bayes; Unimodal.

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1. INTRODUCTION

Let X be a real value random variable (r.v) from the density f(x;θ) and consider the parameter θ as a fixed unknown quantity. If we seek an interval for θ, then it is well known that the standard method for obtaining confidence intervals for θ is the pivotal quantity method. (cf. Huzurbazar (1955), Guenther (1969, 1987), Dahiya and Guttman (1982), Ferentinos (1987, 1988, 1990), Juola (1993), Ferentinos and Kourouklis (1990), Kirmani (1990), Casella and Berger (2002), Rohatgi and Saleh (2001) e.t.c).

Let $Q(X_1, X_2, \ldots, X_n;\theta)$ be a pivotal quantity where $X_1, X_2, \ldots, X_n$ is a random (r.s) from the distribution of f(x;θ). The probability statement

$$P(q_1 < Q < q_2) = 1-\alpha \qquad (1.1)$$

is converted (when possible) to

$$P(q_1^* < \theta < q_2^*) = 1-\alpha\,. \qquad (1.2)$$

If constants q_1, q_2 in (1.1)can be found so that $(q_2^* - q_1^*)$ is minimum, then the interval (q_1^*, q_2^*) is said to be the shortest confidence interval based on Q. Frequently $(q_2^* - q_1^*)$ can be expressed as

$$l = q_2^* - q_1^* = w(x)\varphi(q_1, q_2), \qquad (1.3)$$

where the function w does not involve q_1, q_2 and φ is independent of x. In such situations minimizing $q_2^* - q_1^*$ is the same as minimizing $E(q_2^* - q_1^*)$. On the other hand if constants q_1, q_2 can be determined so that

$$P(Q < q_1) = \alpha/2 \text{ and } P(Q > q_2) = \alpha/2 \qquad (1.4)$$

then the interval (q_1^*, q_2^*) is said to be an equal tails confidence interval.
In both situations we have the same confidence interval, symbolically $CI(q_1^*, q_2^*)$, which is based on the same pivotal quantity Q. What it is different is the determination of the q_1 and q_2 (cf. previous references).

The aim of this work is to clarify and comment on problems that emerge at the process of finding, to investigate the relation of equality of length of these, to point out the non existence of SCI even when the distribution of the pivotal quantity is symmetric and finally to give similar results based on the Bayes approach.

2. MAIN RESULTS

2.1 The case where the SCI coincide with the ETCI

As it was pointed out earlier the SCI and the ETCI differ only as for the determination of q_1, q_2. An interesting question that springs up is when this determination is identical, i.e when those intervals have the same length. And reversely, if the two intervals have the same length does it characterize the distribution of the pivotal quantity? An answer to the last question is given, partially, by the work of Kirmani (1990). In this work it is shown when the ETCI minimize the length based on the pivotal quantity Q. More specifically it shown that "*... an ETCI obtained from a symmetrically distributed pivotal quantity does not necessary have minimum length unless the distribution function of the pivotal quantity is concave on the right of the point of symmetry*". Also (partly) answer in this question gives the theorem 9.3.2 in combination with exercise 9.39 in the book of Casella and Berger

(2002). More concretely if the distribution f(q) of the pivotal quantity Q is unimodal, then the interval $[q_1, q_2]$ that satisfies the relationships

$$\text{(i)} \int_{q_1}^{q_2} f(q)dq = 1-\alpha, \text{ (ii) } f(q_1)=f(q_2)>0 \text{ and (iii) } q_1 \le q^* \le q_2, \tag{2.1}$$

where q* is the median of f(q), is the shortest among all intervals that satisfy (i). Exercise 9.39 says that if f(q) is symmetric and unimodal then for confidence intervals of the form $[q_1, q_2]$ the requirements of theorem 9.3.2 are satisfied and also q_1, q_2 are such that this to be also an ETCI.

The two approaches can be shown to be equivalent if as a point of symmetry we will take the zero point. The precedents give the spark for an overall confrontation of this subject (generalisation and fulfilment) and if the reverse is also true. So we come up with the following proposition.

Proposition 2.1 Let Q=Q(x; θ) be a pivotal quantity with p.d.f. f(q). Let also l_{SCI} and l_{ETCI} be the lengths of a SCI and ETCI, respectively, for θ based on Q. Then, if f(q) is symmetric and unimodal, $l_{SCI}=L_{ETCI}$, provided that the length l is of the form $l=c(q_2-q_1)$, c>0.

Proof:

We define the sequences of points $q_{1,k}$ and $q_{2,k}$ such that

$$\int_{-\infty}^{q_{1,k}} f(q)dq = \alpha/k \text{ and } \int_{q_{2,k}}^{+\infty} f(q)dq = (k-1)\alpha/k, \; k>1. \tag{2.2}$$

Obviously $P(q_{1,k} < Q < q_{2,k}) = 1-\alpha$. Also $F_Q(q_{1,k})=\alpha/k$ and $F_Q(q_{2,k})=1-(k-1)\alpha/k$. Hence

$$q_{1,k} = F_Q^{-1}(\alpha/k) \text{ and } q_{2,k} = F_Q^{-1}(1-(k-1)\alpha/k). \tag{2.3}$$

Since f(q) is symmetric and unimodal, minimizing the length l of the interval $[q_{1,k}, q_{2,k}]$, we get $f(q_{1,k})=f(q_{2,k})$. From this relationship we can determine the values of $q_{1,k}$ and $q_{2,k}$. Without loss of generality we can assume that $-q_{1,k} = q_{2,k}$. Now using (2.3) we get that $-F_Q^{-1}(\alpha/k) = F_Q^{-1}(1-(k-1)\alpha/k)$ which implies that

$$F_Q\left(F_Q^{-1}(\alpha/k)\right) = 1-(k-1)\alpha/k, \tag{2.4}$$

or because of the symmetry of f(q) (see Kirmani 1990)

$$F_Q\left[F_Q^{-1}(\alpha/k)\right] = 1-F_Q\left[F_Q^{-1}(\alpha/k)\right] = 1-\alpha/k. \tag{2.5}$$

From the last two relations we get that $1-\alpha/k=1-(k-1)\alpha/k$ and hence k=2. This completes the proof of the proposition 2.1.

Remarks: (i) We can get a proof of the previous proposition if we combine theorem 9.3.2 and exercise 9.39 of Casella and Berger (2002) or from the theorem of Kirmani (1990). However we believe that the previous proof she is different, sort and at straight line proof.

(ii) Proposition 2.1 has been proved for confidence intervals whose length is of the form $l=c(q_2-q_1)$. However the problem remains unsolved for confidence intervals whose length is of the form $l=c(1/q_1-1/q_2)$. It is the author's guess that if the distribution f(q) is symmetric and unimodal then the only form of the length of the confidence interval is the first one.

(iii) Interest presents the reverse of the proposition 2.1 (because it can constitutes a characterization of f(q)). That is, if $l_{SCI}=l_{ETCI}$ then f(q) is symmetric and unimodal. It is guessed that this may be the case for distributions like normal and t. Although a rigorous proof of it, it is not known (so it remains an opened problem), the following argument supports this idea. "*In the case of the SCI the q_1 and q_2 are one a function of* *the other, i.e $q_1=q_1(q_2)$ (see relation* (1.3)*). This is because the length l must be the shortest one. On the other hand this is not the case for a ETCI. In this case each of the q_1 and q_2 is determined independently of each other (see relations (1.4)). When we say that the two kinds of intervals coincide (i.e $l_{SCI}=l_{ETCI}$) we mean that they are determined by the same q's. That is if q_1^* and q_2^*, ($q_1^* < q_2^*$), are the points which determine the ETCI then $q_1^*=q_1$ and $q_2^*=q_2$ and hence $q_1^*=q_1^*(q_2^*)$. This implies that in the case of ETCI the q's are function of each other and their relation is linear, since the length of the interval is of the form $l=c(q_2-q_1)$. In order this to happen the distribution f(q) must be symmetric and unimodal, i.e. $q_2=q_1+c$".*

(iv) It is known that the SCI based on the pivotal quantity Q it is shortest for the specific pivotal. This means that we can find another pivotal quantity Q* which will give even a shortest interval than that based on Q. (cf. Ferentinos 1988). The question which naturally arises is how to find the pivotal quantity that gives the overall SCI. The literature does not give a clear answer on this point. Intuitively, a reasonable choice is the pivotal quantity to be a function of a sufficient statistic (only), (Guenther, 1969). Moreover, using theorem 9.3.2 of Casella and Berger (2002) we get that the CI $[q_1, q_2]$ is the shortest among all intervals that satisfy (1.1). Now from exercise 9.39 of the same authors, if f(q) is symmetric and unimodal then the previous relation it is satisfied. Thus we can state the following proposition:

Proposition 2.2 The SCI based on pivotal quantities with p.d.f. symmetric and unimodal is the overall shortest confidence interval.

2.2 Monotonicity of f(q) and l(q)

To find a SCI one can use the Lemma 2.1 in Ferentinos and Kourouklis (1990) or equivalently the theorem in Juola (1993). Usually, in most of the cases, one follows the classical minimization process under constraints. This means that one wants to minimize relation (1.3) subject to condition (1.2). The most frequently cases are those where the function $\varphi(q_1, q_2)$ is of the form (q_2-q_1) or $(1/q_1-1/q_2)$. In those cases the minimization problem leads, respectively, to the following relations

$$\text{(i) } f(q_1) = f(q_2) \text{ and (ii) } q_1^2 f(q_1) = q_2^2 f(q_2), \qquad (2.6)$$

or we decide based on the monotonicity of the f(q).

If f(q) is symmetric and unimodal then (w. l. o g. we can assume that $-q_1=q_2$) the quantities q_1 and q_2 are determined from the relation (2.6) (i). However if f(q) is monotonic then it is almost impossible to use relations (2.6). In those cases the minimization problem it is based on the monotonicity of the length l. From this process results the following interest proposition (characterization) for the length l, which depends on the monotonicity of f(q) and facilitates the determination of q_1 and q_2 guiding us to the right direction with respect to the differentiation of q_1 or q_2 (see comment on example 2.2).

Proposition 2.3 Let $Q=Q(x;\theta)$ be a pivotal quantity for a parameter θ with p.d.f. f(q). For the 100(1-α)%CI for θ based on Q of the form $P(q_1<Q<q_2)=1-\alpha$ with length $l_1=w_1(x)(q_2-q_1)$ or $l_2=w_2(x)(1/q_1-1/q_2)$ we have:

(i) if f(q) is a strictly increasing p.d.f. on $[k_1, k_2]$, $k_i \in R$, (i=1, 2) then $l_i(q)$ is strictly decreasing on $[k_1, k_2]$.

(ii)) if f(q) is a strictly decreasing p.d.f. on $[k_1, k_2]$, $k_i \in R$, (i=1, 2) then $l_1(q)$ is strictly increasing on $[k_1, k_2]$.

Proof.

(i) It is easy to see that the minimum of $l_1(q)$ subject to (1.1) occurs for those values of q_1 and q_2 which satisfy the relation

$$\frac{dl_1}{dq_2} = w_1(x)\left(1 - \frac{f(q_2)}{f(q_1)}\right). \tag{2.7}$$

The fact that f(q) is strictly increasing implies that $f(q_1) \neq f(q_2)$. More ever if $q_1 < q_2$ then $f(q_1) < f(q_2)$. Thus from (2.7) and given that $w_1(x) > 0$ we get that $dl_1/dq_2 < 0$. This means that $l_1(q)$ is strictly increasing on some interval $[k_1, k_2]$. Hence the q_1 and q_2, for a SCI, are determined by the relations

$$q_2 = k_2 \text{ and } \int_{q_1}^{k_1} f(q)dq = 1 - \alpha . \tag{2.8}$$

For the case $l_2 = w_2(x)(1/q_1 - 1/q_2)$ we have

$$\frac{dl_2}{dq_2} = w_2(x)\frac{q_1^2 f(q_1) - q_2^2 f(q_2)}{q_1^2 q_2^2 f(q_1)} .$$

Now since $w_2(x) > 0$, $q_1 < q_2$ and $f(q_1) < f(q_2)$ we obtain that $dl_2/dq_2 < 0$. Thus l_2 is strictly decreasing on some interval $[k_1, k_2]$. The q_1 and q_2 can be determined from the relations (2.8).

(ii) Working in a way similar to that in (i) we can show that in the case of l_1 the quantities q_1 and q_2 are determined from the relations

$$q_1 = k_1 \text{ and } \int_{k_1}^{q_2} f(q)dq = 1 - \alpha .$$

In the case of l_2 we can not say anything about the sign of dl_2/dq_1. The quantities q_1 and q_2 are determined from the relation $q_1^2 f(q_1) = q_2^2 f(q_2)$.

Remark: From (i) of the previous proposition we have that when f(q) is strictly increasing then both l_1 and l_2 are strictly decreasing. This means that the SCI (if it exists) take place on the upper point of the interval where Q is defined, that is the point k_2. Hence the derivation of l(q) should be with respect the q_2. In the opposite case the derivation should be with respect the q_1. Another way for expressing the same thing is to set $q_1 = q$ and $q_2 = \delta(q)$ $(q < \delta(q))$.

We will clarify the previous proposition with the following examples.

Example 2.1 (Ferentinos 1990) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with density $f(x,\theta) = g(x)/h(\theta)$, $a(\theta) \leq x \leq b(\theta)$. If $\hat{\theta}$ is a sufficient statistic for θ, then it is known that the quantity $Q = h(\hat{\theta})/h(\theta)$ is a pivotal quantity with distribution (Huzurbazar 1955) $f(q) = nq^{n-1}$, $0 \leq q \leq 1$. Obviously f(q) is strictly increasing on [0, 1] for n>1. The CI based on Q can be found from the relation $P(q_1 < Q, q_2) = 1 - \alpha$, from which we get that $l = h(\hat{\theta})(1/q_1 - 1/q_2)$. So, from proposition 2.2, the length of the interval l is strictly decreasing and hence the SCI is obtained on the points $q_2 = 1$ and q_1 given from the relation $\int_{q_1}^{1} f(q)dq = 1 - \alpha$. Finally the SCI for $h(\theta)$ we get is the well known one $h(\hat{\theta}) \leq h(\theta) \leq h(\hat{\theta})\alpha^{-1/n}$.

Example 2.2 (Guenther 1969) Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $f(x, \theta)=e^{-(x-\theta)}$, $x \geq \theta$. If $T=\min X_i$ ($i=1, 2, \ldots, n$) is the sufficient statistic for the parameter θ then $Q=2n(T-\theta)$ is a pivotal quantity with p.d.f $f(q)=(1/2)e^{-q/2}$, $q \geq 0$. It is clear that $f(q)$ is strictly decreasing on $[0, \infty]$ and hence, according to proposition 2.2 (ii), $l(q)$ is strictly increasing on $[0, \infty]$. Thus the SCI will be given from the points q_1 and q_2, where $q_1=0$ and q_2 is determined from the relation $\int_0^{q_2} f(q)dq = 1-\alpha$. Finally the SCI is the $(T+\ln\alpha/n, T)$.
Note that if we differentiate with respect the q_2, then the length is still a strictly increasing function, but we can not get $q_2=0$ since $q \in [0, \infty]$ and $q_1<q_2$. Thus we have to differentiate with respect to q_1.

2.3 The case where a SCI does not always exist

The SCI does not always exist even when the distribution of the pivotal quantity $f(q)$ is symmetric. At this point it is worth to comment and make widely known two examples given by Kirmani (1990).

Example 2.3 Let X have the density $f(x, \theta)=|x-\theta|$, $\theta-1<x<\theta+1$, $-\infty<\theta<+\infty$. The quantity $Q=X-\theta$ has the symmetric distribution $f(q)=|q|$ $-1 \leq q \leq 1$ and is a pivotal one. To find a SCI or a ETCI we use the relation (1.1). At the moment will discuss the case where $-1<q_1<0<q_2<1$. The cases $0<q_1<q_2<1$ and $-1<q_1<q_2<0$ give us CIs whose level of significance is less than 50% since in both cases $½<\alpha<1$.So for the case $-1<q_1<0<q_2<1$ we have: $P(x-q_2 \leq \theta \leq x-q_1)=1-\alpha$ and the interval for θ has length $l=q_2-q_1$. Minimizing this length subject to (1.1) gives $f(q_1)=f(q_2)$ and hence $-q_1=q_2$. From that we get that $\left.\frac{d^2 l}{dq_1^2}\right|_{q_1=-q_2} < 0$ and hence $l(q)$ can not be minimized (actually is maximized) which means that a SCI does not exist in this case. On the contrary an ETCI exists and has the form $[x-(1-\alpha)^{1/2}, x+(1-\alpha)^{1/2}]$.
At this point we have to say that the proposition 2.1 can not be applied since the density $f(q)$ it is not unimodal.

If we want a SCI or an ETCI for theoretical reasons and not for practical use, we can work out the case $0<q_1<q_2<1$. In this case $f(q)$ is strictly increasing and making use of proposition 2.3 we get that the SCI is of the form $[x-1, x-(2\alpha-1)^{1/2}]$ whereas the ETCI has the form $[x-(1-\alpha)^{1/2}, x+\alpha^{1/2}]$.

Example 2.4 Let X have density $f(x,\theta)=(1/2\theta)e^{-|x|/\theta}$, $-\infty<x<+\infty$, $\theta>0$. The quantity $Q=X/\theta$ is a pivotal quantity with density $f(q)=.5e^{-|q|}$, $-\infty<q<+\infty$. As in the previous example the most interesting case is the case where $-\infty<q_1<0<q_2<+\infty$. The CI we get, based on the previous pivotal quantity, has the form $[\max(x/q_1, x/q_2), +\infty]$. Obviously its length l equals to infinity ($l=\infty$). This implies that there is no meaning to search for a SCI. On the contrary an ETCI can easily be obtained and has the form $[\max(x/\ln\alpha, -x/\ln\alpha), +\infty]$.

Let's now consider the quantity $Q^*=2|x|/\theta$. It can be shown that it is a pivotal quantity with p.d.f. $f(q^*)=.5e^{-q^*/2}$, $q^*>0$. The CI based on this quantity takes the form$\left(\frac{2|x|}{q_2}, \frac{2|x|}{q_1}\right)$. Since f(q*) is decreasing the q_1 and q_2 will be determined from the relation $q_1^2 f(q_1) = q_2^2 f(q_2)$ (see proposition 2.3).

2.4 Bayes approach

Although some textbooks in Mathematical Statistics discuss Bayes confidence intervals (BCI), the concept of a Bayes shortest confidence interval (BSCI) commands little or no attention. The term is mentioned in Rohatgi and Saleh (2001), Casella and Berger (2002), Beaumont (1980), Mood et all (1974) and Silvey (1975). However, neither text offers any further discussion of the topic.

Let X be a r.v having a density $f(x|\theta)$. Suppose that $\pi(\theta)$ is a prior distribution of θ and $\pi(\theta|x)$ is the posterior distribution corresponding to $f(x|\theta)$ and $\pi(\theta)$. Given $\pi(\theta|x)$ the $100(1-\alpha)\%$ BCI for θ is defined by

$$P(q_1<(\theta|x)<q_2)=1-\alpha \quad \text{or} \quad \int_{q_1}^{q_2} \pi(\theta \mid x)d\theta = 1-\alpha . \qquad (2.9)$$

Hence, in order to obtain a BSCI for θ, we need to choose q_1, q_2 such that the length

$$l=q_2-q_1 \qquad (2.10)$$

is minimum under the condition (2.9). In the case where $\pi(\theta|x)$ is symmetric and unimodal then q_1 and q_2 can be determined from the relation $\pi(q_1|x)=\pi(q_2|x)$. In a different case we have to exam the monotonicity of $l(q)$. In the last case the proposition 2.3 can be used without any restriction since the form of l is always of the form (q_2-q_1). In the Bayes approach θ is a r.v. and in general the posterior probability $\pi(\theta|x)$ can be considered as a pivotal quantity, in the sense that it is a function of θ and x has some "known" distribution. After that we can state the following proposition.

Proposition 2.4 If $\pi(\theta|x)$ is the posterior p.d.f. of $\theta|x$, then for the BCI of θ of the form (2.9) and length (2.10) we have that:

i) if $\pi(\theta|x)$ is a strictly increasing p.d.f on $[k_1, k_2]$, $k_i \in R$ $(i=1, 2)$, then $l(q)$ is strictly decreasing on $[k_1, k_2]$.

ii) if $\pi(\theta|x)$ is a strictly decreasing p.d.f on $[k_1, k_2]$, $k_i \in R$ $(i=1, 2)$, then $l(q)$ is strictly increasing on $[k_1, k_2]$.

Those results, as in the classical case, make easier the determination of q_1 and q_2.

Remarks: (i) In the present case theorem 9.3.2 of Casella and Berger is valid without any comment (like those made for the classical case) because the length l is always of the form q_2-q_1. (See also Casella and Berger corollary 9.3.10).

(ii) A BETCI can be defined from the relations

$$\int_{-\infty}^{q_1} \pi(\theta \mid x)d\theta = \alpha/2 \quad \text{and} \quad \int_{q_2}^{+\infty} \pi(\theta \mid x)d\theta = \alpha/2 \quad .$$

In this case proposition 2.1 is always true , i. e. if $\pi(\theta|x)$ is symmetric and unimodal then $l_{BSCI}=l_{BETCI}$. The comments made for a similar remark in the classical case are still true.

(iii) Since the determination of q_1 and q_2 is based on the posterior p.d.f., $\pi(\theta|x)$, many authors (see e.g Bickel and Doksum (2001), Casella and Berger (2002)), in order to distinguish between classical and Bayesian confidence sets, they use the term credible sets for the second case.

In many cases the BSCI for a parameter θ has shorter length than the corresponding SCI in the classical case. This it is maybe expected since in the Bayesian approach we have more information about the parameter θ.

We demonstrate the previous discussion with the following examples.

Example 2.5 Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\theta, 1)$ and let the prior distribution of θ be the $N(0, 1)$. It is well known (see Mood et all 1974, Bickel and Doksum 2001) that the posterior distribution of θ, $\pi(\theta|x)$, is the $N(n\bar{X}/(n+1),\ 1/(n+1))$. Since $\pi(\theta|x)$ is symmetric and unimodal, by previous discussion, the q_1 and q_2 will be found from the relation $\pi(q_1|x)=\pi(q_2|x)$ or $(q_1 - n\bar{X}/(n+1))^2 = (q_2 - n\bar{X}/(n+1))^2$, which implies that $q_1 = 2n\bar{X}/(n+1) - q_2$. Combining it with the relation $P(\theta|x \geq q_2)=\alpha/2$ we get that $q_1 = \frac{n\bar{X}}{n+1} - \frac{1}{\sqrt{n+1}} z_{\alpha/2}$ and hence $q_2 = \frac{n\bar{X}}{n+1} + \frac{1}{\sqrt{n+1}} z_{\alpha/2}$. Thus we get the well known CI which is the shortest. The BETCI are found using the usual relationships. Note that in this case the reverse of proposition 2.1 is also true.

Example 2.6 Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution $U(0, \theta)$ and let the prior distribution of θ be the Pareto with density $\pi(\theta) = kx_0^k/\theta^{k+1}$, $x_0 \leq \theta < \infty$, where x_o and k are known quantities. It can be shown that

$$\pi(\theta \mid x) = \frac{(n+k)(X^*_{(n)})^{n+k}}{\theta^{n+k+1}},\quad X^*_{(n)} \leq \theta < \infty,$$

where $X^*_{(n)} = \max(X_{(n)}, x_o)$ and $X_{(n)}=\max X_i$. Since $\pi(\theta|x)$ is strictly decreasing on $[X^*_{(n)}, \infty]$, $l(q)$ is strictly increasing on the same interval and hence $q_1 = X^*_{(n)}$ and $q_2 = X^*_{(n)}\alpha^{-\frac{1}{n+k}}$. Thus the BSCI for θ is the $\left(X^*_{(n)},\ X^*_{(n)}\alpha^{-\frac{1}{n+k}}\right)$.

Example 2.7 Let X be a r.v with p.d.f. $f(x, \theta)=e^{-(x-\theta)}$, $-\infty<\theta\leq x<\infty$, and let the prior p.d.f. of θ be the $\pi(\theta)=\theta e^{-\theta}$, $\theta\geq 0$. Here $\pi(\theta|x)=2\theta/x^2$, $0\leq\theta\leq x$. Now, $\pi(\theta|x)$ is strictly increasing on $[0, x]$ which means that $l(q)$ is strictly decreasing on the same interval. Thus the minimum of $l(q)$ occurs at $q_2=x$ and $q_1=x\alpha^{1/2}$, i.e the BSCI is the $(x\alpha^{1/2}, x)$.

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